Noncommutative Schur polynomials and the crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$-vertex model

Christian Korff
School of Mathematics and Statistics, University of Glasgow
15 University Gardens, Glasgow G12 8QW, Scotland, UK
E-mail: c.korff@maths.gla.ac.uk

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Abstract

Starting from the Verma module of $U_q\mathfrak{sl}(2)$ we consider the evaluation module for affine $U_q\widehat{\mathfrak{sl}}(2)$ and discuss its crystal limit ($q \to 0$). There exists an associated integrable statistical mechanics model on a square lattice defined in terms of vertex configurations. Its transfer matrix is the generating function for noncommutative *complete* symmetric polynomials in the generators of the affine plactic algebra, an extension of the finite plactic algebra first discussed by Lascoux and Schützenberger. The corresponding noncommutative *elementary* symmetric polynomials were recently shown to be generated by the transfer matrix of the so-called phase model discussed by Bogoliubov, Izergin and Kitanine. Here we establish that both generating functions satisfy Baxter's TQ-equation in the crystal limit by tying them to special $U_q\widehat{\mathfrak{sl}}(2)$ solutions of the Yang-Baxter equation. The TQ-equation amounts to the well-known Jacobi-Trudi formula leading naturally to the definition of noncommutative Schur polynomials. The latter can be employed to define a ring which has applications in conformal field theory and enumerative geometry: it is isomorphic to the fusion ring of the $\widehat{\mathfrak{sl}}(n)_k$ -WZNW model whose structure constants are the dimensions of spaces of generalized θ-functions over the Riemann sphere with three punctures.

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1 Introduction

Integrable systems have many connections with different areas in pure mathematics. In this article we shall focus on combinatorial aspects of a particular quantum integrable system, the exactly solvable, statistical vertex model associated with the quantum affine algebra $U_q\widehat{\mathfrak{sl}}(2)$ and "infinite" spin. For spin 1/2 this model specialises to the the well-known six-vertex model or XXZ quantum

spin-chain. By taking the crystal limit [15] ($q \to 0$) one arrives at a drastically simplified version of this model exhibiting nice combinatorial features: the Takahashi-Satsuma cellular automaton [20] (or box and ball system); see also e.g. [8], [10] and references therein for the case of higher (finite) spin and rank.

In the case of infinite spin, i.e. each site of the chain now carries an infinite-dimensional representation of $U_q\widehat{\mathfrak{sl}}(2)$ instead of a finite-dimensional one, there exists a link with enumerative geometry: the commuting transfer matrices generate a ring of symmetric polynomials in a noncommutative alphabet, the generators of the *affine* plactic algebra, whose *finite* version has been introduced by Lascoux and Schützenberger [17] (see also [7] for a discussion of the *finite* plactic algebra in the context of noncommutative Schur polynomials). It was shown in [16, Part I] that the noncommutative Schur polynomials related to the affine plactic algebra can be employed to define a ring which is isomorphic to the fusion ring of the $\widehat{\mathfrak{sl}}(n)_k$ Wess-Zumino-Novikov-Witten (WZNW) model, a conformal field theory (CFT) with particularly nice algebraic and geometric aspects. Here $k \geq 0$ is a non-negative integer called the "level". The fusion ring is one of the essential data of a CFT [5] and in the case of the WZNW model its structure constants coincide with the dimensions of spaces of generalized θ-functions over the Riemann sphere with three punctures; see e.g. [2].

In this article we shall discuss how the combinatorial description of the fusion ring presented in [16, Part I] is obtained from the crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$ model with infinite spin. In Section 2 we demonstrate on a simple example how under the action of the affine plactic algebra the state space of the model decomposes into nice lattices, called "crystals" which depend on the level k and can be described in terms of coloured directed graphs (usually called crystal graphs) related to the quantum affine algebra $U_q\widehat{\mathfrak{sl}}(n)$ with $n > 2$ being the number of lattice sites. Section 3 contains the definition of the $U_q\widehat{\mathfrak{sl}}(2)$ vertex model with infinite spin in the crystal limit. In particular we show in Section 4 that the generating function of the *noncommutative complete symmetric polynomials* coincides with its transfer matrix; see Proposition 4.1 in the text. This new result complements the discussion in [16, Part I, Section 4,5] where the transfer matrix of the phase model, first introduced by Bogoliubov, Izergin and Kitanine [3], has been identified with the generating function of the *noncommutative elementary symmetric polynomials* [16, Prop 5.13]. As in the ring of symmetric functions over commutative variables, also in the present case both sets of polynomials in noncommutative variables are linked via a determinant formula (a special case of the Jacobi-Trudi identity) for which we shall state an alternative proof to the one given in [16, Def 5.10 and Cor 6.9]. Namely, employing the relation to the $U_q\widehat{\mathfrak{sl}}(2)$ algebra we show that the transfer matrix of the $U_q\widehat{\mathfrak{sl}}(2)$ model with infinite spin and the transfer matrix of the phase model obey Baxter's famous TQ-equation [1] in the crystal limit. In the concluding section we summarise the connection with the WZNW fusion ring, giving a brief account of some of the main results from [16, Part I], and state a novel recursion formula for fusion coefficients.

2 The crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$-Verma module

We set out by introducing the central algebraic structure: we recall the definition of the quantum affine algebra $U_q\widehat{\mathfrak{sl}}(2)$ and then consider a particular infinite-dimensional representation of it.

Definition 2.1 *The q-deformed universal enveloping algebra $U_q\widehat{\mathfrak{sl}}(2)$ is the unital associative algebra over $\mathbb{C}(q)$ generated from the letters $\{E_i, F_i, K_i^{\pm 1}\}_{i=0,1}$ subject to the algebraic relations*

$$K_i E_j K_i^{-1} = q^{A_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-A_{ij}} F_j \;, \tag{2.1}$$
$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \quad K_i K_j = K_j K_i,$$

and for $i \neq j$

$$\sum_{p=0}^{1-A_{ij}} (-1)^p \begin{bmatrix} 1 - A_{ij} \\ p \end{bmatrix}_q X_i^{1-A_{ij}-p} X_j X_i^p = 0, \qquad X_i = E_i, F_i, \tag{2.2}$$

where $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ is the Cartan matrix of $\widehat{\mathfrak{sl}}(2)$ and we have set $[x]_q = (q^x - q^{-x})/(q - q^{-1})$, $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q := \frac{[m]_q!}{[n]_q![m-n]_q!}$, $[n]_q! := \prod_{k=1}^n [k]_q$ as usual. We denote by $U_q\mathfrak{sl}(2) \subset U_q\widehat{\mathfrak{sl}}(2)$ the subalgebra generated from $\{E_1, F_1, K_1^{\pm 1}\}$.

We introduce the following coproduct $\Delta : U_q\widehat{\mathfrak{sl}}(2) \to U_q\widehat{\mathfrak{sl}}(2) \otimes U_q\widehat{\mathfrak{sl}}(2)$,

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \;\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \;\Delta(K_i) = K_i \otimes K_i \;. \tag{2.3}$$

It is well-known that $U_q\widehat{\mathfrak{sl}}(2)$ can be turned into a Hopf algebra by defining in addition a counit and antipode; see e.g. [6, 11]. However, as we shall not use the latter maps here, we omit their definition. We now consider a particular module of $U_q\widehat{\mathfrak{sl}}(2)$ which we shall use throughout this article, first to define various other related algebras in the crystal limit and then to define certain statistical mechanics models.

First we recall the definition of a Verma module for the finite subalgebra $U_q\mathfrak{sl}(2)$. Let $\mathcal{I}_\mu \subset U_q\mathfrak{sl}(2)$ be the left ideal generated by E_1 and $K_1 - \mu q^{-1} 1$. Consider the module $M_\mu = U_q\mathfrak{sl}(2)/\mathcal{I}_\mu$ which is free over the subalgebra generated by F_1 (by a quantum version of the Poincaré-Birkhoff-Witt theorem). Set $v_0 = 1 + \mathcal{I}_\mu$ and $v_m := F_1^m v_0$ for all $m \in \mathbb{N}$, then $\{v_m\}_{m \in \mathbb{Z}_{\geq 0}}$ is a basis and the following relations hold

$$\begin{aligned} K_1 v_m &= \mu q^{-2m-1} v_m, \qquad F_1 v_m = v_{m+1}, \\ E_1^m v_0 &= 0, \qquad E_1 v_m = \left(\tfrac{\mu q^{-m} - \mu^{-1} q^m}{q - q^{-1}}\right) [m]_q v_{m-1} \;. \end{aligned} \tag{2.4}$$

Consider $M_\mu(u) = \mathbb{C}(q)[u, u^{-1}] \otimes_{\mathbb{C}(q)} M_\mu$ then we can regard $M_\mu(u)$ as a $U_q\widehat{\mathfrak{sl}}(2)$-module by defining the following action of the affine generators

$$E_0 \to u^{-1} \otimes F_1, \qquad F_0 \to u \otimes E_1, \qquad K_0^{\pm 1} \to 1 \otimes K_1^{\mp 1} \tag{2.5}$$

while keeping the action of the non-affine generators unchanged, $X_1 \to 1 \otimes X_1$ with $X_1 = E_1, F_1, K_1^{\pm 1}$. For $\upsilon \in \mathbb{C}^\times$ the corresponding evaluation module is the one obtained by taking the quotient in $\mathbb{C}(q)[u, u^{-1}]$ with respect to the maximal ideal generated by $u - \upsilon$.

Remark 2.2 *For generic values of μ this module is known to be irreducible. If we choose $\mu = q^d$ for some positive integer d, we have $E_1^d v_d = 0$ and we can identify the submodule spanned by the first d vectors $v_0, v_1, \ldots, v_{d-1}$ with the standard $U_q\mathfrak{sl}(2)$ module of dimension d upon imposing the condition $F_1^m = 0$ for $m \geq d$. In particular we recover the fundamental representation for $d = 2$.*

2.1 Crystal limit and phase maps

We have defined the quantum algebra $U_q\widehat{\mathfrak{sl}}(2)$ over the ring of rational functions in the indeterminate q. Naively speaking we now wish to take the limit $q \to 0$, which is referred to as the "crystal limit" as it was originally considered in the context of statistical mechanics models, where it corresponds to the low temperature limit when the system "crystallizes" into a single configuration. In more technical terms we restrict the algebra into what follows to the ring of regular functions in q and then take the quotient with respect to ideal generated by q. We shall refer to this procedure as the crystal limit in accordance with the literature; see for example [11, Section 4.2, page 67] and references therein. Our main interest are the combinatorial features which emerge in this limit.

Proposition 2.3 *Denote by $\mathbb{A} \subset \mathbb{C}(q)$ the ring of functions which are regular at $q = 0$ and let $\mathcal{L} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{A} v_m$. Then*

(i) $\mathcal{L}$ generates M_μ as a vector space over $\mathbb{C}(q)$, $M_\mu = \mathbb{C}(q) \otimes_{\mathbb{A}} \mathcal{L}$.

(ii) Define $\tilde{E}_1 = -q^{-1} K_1^{-1} E_1$ then $\tilde{E}_1 \mathcal{L} \subset \mathcal{L}$ and $F_1 \mathcal{L} \subset \mathcal{L}$.

Proof. The first assertion is obvious. The statements under (ii) are easily verified from (2.4). For instance, we find that

$$\tilde{E}_1 v_m = \left[\left(\frac{1 - \mu^{-2} q^{2m}}{1 - q^2} \right) \left(\frac{q^{2m} - 1}{q^2 - 1} \right) \right] v_{m-1} \tag{2.6}$$

where the coefficient is obviously regular at $q = 0$. Similarly, we compute the relation

$$\left[\tilde{E}_1 F_1 - q^2 F_1 \tilde{E}_1 \right] v_m = \left[\frac{1 - \mu^{-2} q^{4m+2}}{1 - q^2} \right] v_m \tag{2.7}$$

which we will use below. ■

In what follows we consider the crystal limit of M_μ, that is we will consider $\mathcal{M} = \mathcal{L}/q\mathcal{L}$ as $\mathbb{C}$-vector space in the natural way. From (2.4) and (2.6) we infer that the quantum algebra elements K_1, $\tilde{E}_1$ and F_1 then induce the following maps $N, \varphi, \varphi^* : \mathcal{M} \to \mathcal{M}$ defined via

$$N v_m = m v_m, \quad \varphi^* v_m = v_{m+1} \quad \text{and} \quad \varphi v_m := \begin{cases} 0, & m = 0 \\ v_{m-1}, & m > 0 \end{cases} . \tag{2.8}$$

In particular, we have the following crystal limit of the $U_q\mathfrak{sl}(2)$ relations (see equations (2.7)),

$$\varphi\varphi^* = 1 \qquad \text{and} \qquad \varphi^*\varphi v_m = \begin{cases} 0, & m = 0 \\ v_m, & m > 0 \end{cases} . \tag{2.9}$$

Fix an integer $n > 2$ and consider the tensor product $\mathcal{H} = \mathcal{M}^{\otimes n}$. We extend the maps (2.8) to $\mathcal{H} = \mathcal{M}^{\otimes n}$ by defining for $i = 1, \ldots, n$

$$\varphi_i := 1 \otimes \cdots \otimes \underset{i}{\varphi} \otimes \cdots \otimes 1 \tag{2.10}$$

and similarly, $\varphi_i^* := 1 \otimes \cdots \otimes \underset{i}{\varphi^*} \otimes \cdots \otimes 1$, $N_i := 1 \otimes \cdots \otimes \underset{i}{N} \otimes \cdots \otimes 1$. We obtain the phase algebra discussed by Bogoliubov, Izergin and Kitanine in [3]; see also references [5], [6] and [8] therein. The proof of the following result can be found in [16, Prop 3.1].

Proposition 2.4 *The φ_i, φ_i^* and N_i generate a subalgebra $\hat{\Phi}$ of $\mathrm{End}(\mathcal{H})$ which can be realized as the algebra Φ with the following generators and relations for $1 \le i, j \le n$:*

$$\varphi_i\varphi_j = \varphi_j\varphi_i, \quad \varphi_i^*\varphi_j^* = \varphi_j^*\varphi_i^*, \quad N_iN_j = N_jN_i \tag{2.11}$$

$$N_i\varphi_j - \varphi_jN_i = -\delta_{ij}\varphi_i, \quad N_i\varphi_j^* - \varphi_j^*N_i = \delta_{ij}\varphi_i^*, \tag{2.12}$$

$$\varphi_i\varphi_i^* = 1, \quad \varphi_i\varphi_j^* = \varphi_j^*\varphi_i \ \text{ if } \ i \neq j, \tag{2.13}$$

$$N_i(1 - \varphi_i^*\varphi_i) = 0 = (1 - \varphi_i^*\varphi_i)N_i. \tag{2.14}$$

If we introduce the scalar product on the vector space $\mathcal{H}$ by

$$\langle \alpha v_{m_1} \otimes \cdots \otimes v_{m_n}, \beta v_{m_1'} \otimes \cdots \otimes v_{m_n'} \rangle = \overline{\alpha}\beta \prod_{i=0}^{n-1} \delta_{m_i, m_i'},$$

for $\alpha, \beta \in \mathbb{C}$, then $\langle \varphi_i^* v, v' \rangle = \langle v, \varphi_i v' \rangle$ for any $v, v' \in \mathcal{H}$.

2.2 Crystallisation of the state space

We now decompose the tensor product $\mathcal{H} = \mathcal{M}^{\otimes n}$ into an infinite direct sum,

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}_{\ge 0}} \mathcal{H}_k, \qquad \mathcal{H}_k = \mathbb{C}\left\{ v_{m_1} \otimes \cdots \otimes v_{m_n} \;\middle|\; \sum_{i=1}^n m_i = k \right\}, \tag{2.15}$$

where we set $\mathcal{H}_0 = \mathbb{C}\{v_0 \otimes \cdots \otimes v_0\} \cong \mathbb{C}$ and the summation index k is the "level" of the WZNW model. For notational convenience we will often identify a basis vector $v_{m_1} \otimes \cdots \otimes v_{m_n}$ in $\mathcal{H}$ with the composition $\boldsymbol{m} = (m_1, \ldots, m_n)$ or, equivalently, the partition $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_n)$ whose associated Young diagram contains m_i columns of height i. We denote the corresponding set of such partitions by P^+ and the subset corresponding to $\mathcal{H}_k$ by P_k^+. Obviously each $\hat{\lambda} \in P_k^+$ has at most n parts and $\hat{\lambda}_1 = k$. We now wish to explain how the subspaces $\mathcal{H}_k$ can be identified with crystal graphs of the affine quantum algebra $U_q\widehat{\mathfrak{sl}}(n)$.

Definition 2.5 *Let* $\mathcal{A} = \{a_0, a_1, a_2, \ldots a_{n-1}\}$. *The* local affine plactic algebra $\mathrm{Pl} = \mathrm{Pl}(\mathcal{A})$ *is the free algebra generated by the elements of* $\mathcal{A}$ *modulo the relations*

$$a_i a_j - a_j a_i = 0, \qquad \textit{if } |i-j| \neq 1 \mod n, \tag{2.16}$$

$$a_{i+1} a_i^2 = a_i a_{i+1} a_i, \qquad a_{i+1}^2 a_i = a_{i+1} a_i a_{i+1}, \tag{2.17}$$

where in (2.17) *all variables are understood as elements in* $\mathcal{A}$ *by taking indices modulo* n. *Let* $\mathrm{Pl}_{\mathrm{fin}} = \mathrm{Pl}_{\mathrm{fin}}(\mathcal{A}')$ *denote the* local finite plactic algebra *generated from* $\mathcal{A}' = \{a_1, a_2, \ldots a_{n-1}\}$*; compare with [7].*

We recall the following result from [16, Prop 5.8]:

Proposition 2.6 *There is a homomorphism of algebras* $\mathrm{Pl}_{\mathrm{fin}} \to \Phi$ *such that*

$$a_j \mapsto \varphi_{j+1}^* \varphi_j, \quad j = 1, ..., n-1, \tag{2.18}$$

hence, the representation of the phase algebra Φ *given by (2.8) and (2.10) lifts to a representation of the local plactic algebra* $\mathrm{Pl}_{\mathrm{fin}}$. *Mapping* $a_0 = a_n$ *to* $z\varphi_1^* \varphi_n$ *it lifts in addition to a representation of* Pl *on* $\mathcal{H}[z] = \mathbb{C}(z) \otimes_{\mathbb{C}} \mathcal{H}$ *with* z *an indeterminate. Both representations are faithful.*

As mentioned in [16, Remark 5.9] the subspace $\mathcal{H}_k \subset \mathcal{H} = \mathcal{M}^{\otimes n}$ together with the action (2.18) of the local affine plactic algebra can be identified with the crystal graph of the k^{th}-symmetric tensor representation of the vector representation of $U_q\widehat{\mathfrak{sl}}(n)$ [12]. In the literature this crystal graph is also known as the affinization of the Kirillov-Reshetikhin crystal graph $\mathfrak{B}_{1,k}$ of type A; see e.g. [19, Section 3]. We discuss an explicit example below.

Definition 2.7 *The quantum universal enveloping algebra* $U_q\widehat{\mathfrak{sl}}(n)$, $n > 2$ *is the associative unital* $\mathbb{C}(q)$*-algebra generated by* $\{E_i, F_i, K_i^{\pm 1}\}_{i=0}^{n-1}$ *subject to the analogous identities as in (2.1), (2.2) but with respect to the* $\widehat{\mathfrak{sl}}(n)$ *Cartan matrix,*

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix} .$$

We recall that the vector representation $V = \mathbb{C}\{v_1, ..., v_n\}$ associated with the fundamental weight ω_1 is given by

$$E_i v_r = \delta_{i,r-1} v_{r-1}, \qquad F_i v_r = \delta_{r,i} v_{r+1}, \qquad K_i v_r = q^{\delta_{i,r} - \delta_{i,r-1}} v_r \, . \tag{2.19}$$

As in the case of $n = 2$ this $U_q\mathfrak{sl}(n)$-module can be turned into an evaluation module $V(z)$ for any $z \in \mathbb{C}^\times$ by setting

$$E_0 v_r = z^{-1} \delta_{r,1} v_n, \qquad F_0 v_r = z \; \delta_{r,n} v_1, \qquad K_0 v_r = q^{\delta_{r,n} - \delta_{r,1}} v_r \, . \tag{2.20}$$

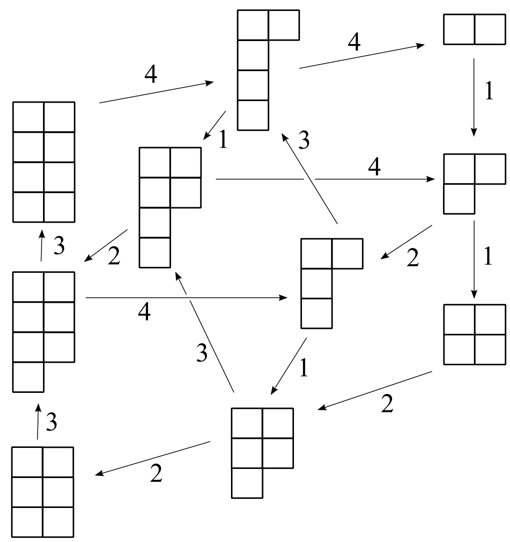


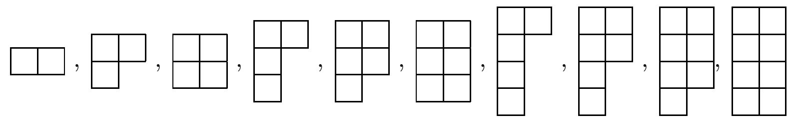
Figure 2.1: The crystal graph for $n = 4, k = 2$ and $z = 1$. The vertices are the elements in P_k^+. Two vertices $\hat{\lambda}, \hat{\mu}$ are connected via an edge of colour i, if $\hat{\mu} = a_i\hat{\lambda}$.

Consider the tensor product $V^{\otimes k} \cong V(zq^{-k+1}) \otimes V(zq^{-k+3}) \otimes \cdots \otimes V(zq^{k-1})$ and let $V_{n,k}$ denote the subspace invariant under the natural action of the Hecke algebra on $V^{\otimes k}$; see e.g. [6, Chapter 10.2] and references therein. There is a distinguished basis $\mathcal{B}_{n,k} = \{v_{\hat{\lambda}}\}_{\hat{\lambda} \in P_k^+} \subset V_{n,k}$ such that the pair $(\mathcal{L}_{n,k}, \mathcal{B}_{n,k})$ with $\mathcal{L}_{n,k} = \bigoplus_{\hat{\lambda} \in P_k^+} \mathbb{A} v_{\hat{\lambda}}$ forms a crystal basis of $V_{n,k}$ and the plactic generators a_i^*, a_i coincide with Kashiwara's crystal operators $\tilde{E}_i, \tilde{F}_i$, respectively. (We refer the reader to [11] for an explanation of these terms.) For instance, let $\hat{\lambda} \in P_k^+$ be a strict partition, all parts are mutually distinct, and denote by $\hat{\lambda}^t$ its transpose. Then the corresponding basis vector in $V^{\otimes k}$ is given by

$$v_{\hat{\lambda}} = \frac{1}{[k]_q!} \sum_{\sigma \in S_k} q^{-\ell(\sigma)} v_{\hat{\lambda}^t_{\sigma(k)}} \otimes \cdots \otimes v_{\hat{\lambda}^t_{\sigma(1)}} .$$

In particular, $\mathcal{B}_{n,k} \cong P_k^+$ are identical as sets and under the action of the local affine plactic algebra P_k^+ can be viewed as a coloured, oriented graph (called the crystal graph of $V_{n,k}$). The vertices of this graph are the elements in P_k^+ and the edges of colour i, $\hat{\lambda} \xrightarrow{i} \hat{\mu}$, are given by the relation $\hat{\mu} = a_i\hat{\lambda}$, where a_i adds a box in the $(i+1)^{\text{th}}$ row (if allowed) for $i = 1, \ldots, n-1$. The letter a_n removes a column of height n and adds a box in the first row if possible.

Example 2.8 *Let* $n = 4,\ k = 2$ *and for simplicity set* $z = 1$*. Then*

$P_k^+ = \Big\{$  $\Big\}$

The corresponding basis vectors in $V^{\otimes k}$ *for the first 5 elements are*

$$v_1 \otimes v_1,\ \frac{v_2\otimes v_1 - q^{-1} v_1 \otimes v_2}{[2]_q},\ v_2 \otimes v_2,\ \frac{v_3\otimes v_1 - q^{-1} v_1 \otimes v_3}{[2]_q},\ \frac{v_3\otimes v_2 - q^{-1} v_2 \otimes v_3}{[2]_q}, \ldots$$

et cetera. The crystal graph resulting from the action of the local affine plactic algebra on P_k^+ *is depicted in Figure 2.1 for* $z = 1$*.*

Note the difference in role played by the local affine plactic algebra and the phase algebra. While the plactic algebra $\mathrm{Pl}(\mathcal{A})$ preserves the level k and describes the crystal structure of $\mathcal{H}_k$ with respect to the quantum affine algebra $U_q\widehat{\mathfrak{sl}}(n)$, the phase algebra $\hat{\Phi}$ increases and decreases the level k, where the maps $\varphi_i, \varphi_i^* : \mathcal{H}_k \to \mathcal{H}_{k\mp 1}$ correspond to the crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$ generators. They generate a "tower" of crystals, the simplest example, $n = 3$, is depicted in Figure 2.2.

3 R-matrices in the crystal limit

We are now going to define an integrable vertex model on $\mathcal{H} = \mathcal{M}^{\otimes n}$ along the same lines as the models discussed in [1]. In the appendix it is shown that it arises from taking the crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$ intertwiner, generically called R-matrix, for the tensor product $M_\mu(u) \otimes M_\nu(v)$ of the module discussed above.

3.1 Definition of the vertex model

Consider a $n \times n'$ square lattice with periodic boundary conditions in the horizontal direction. On the edges of the square lattice live statistical variables $m \in \mathbb{Z}_{\geq 0}$, which we identify with the basis vectors $\{v_m\}_{m\in\mathbb{Z}_{\geq 0}}$ in $\mathcal{M}$. Then a row configuration in the lattice, i.e. an assignment of statistical variables $\boldsymbol{m} = (m_1, \ldots, m_n)$ along one row of vertical edges, fixes a vector $v_{m_1} \otimes \cdots \otimes v_{m_n} \in \mathcal{H} = \mathcal{M}^{\otimes n}$. Similarly, a fixed lattice configuration of the entire lattice can be seen as a vector in $\mathcal{H}^{\otimes n'}$. Not each lattice configuration is allowed, we single out particular ones by assigning to each local configuration around a single vertex a "Boltzmann weight" (a pseudo-probability)

$$\mathcal{R}^{a,b}_{c,d}(u) = \begin{cases} u^a, & d = a + b - c,\ b \geq c \\ 0, & \text{else} \end{cases}, \tag{3.1}$$

where $a, b, c, d \in \mathbb{Z}_{\geq 0}$ are the statistical variables; see Figure 3.1.

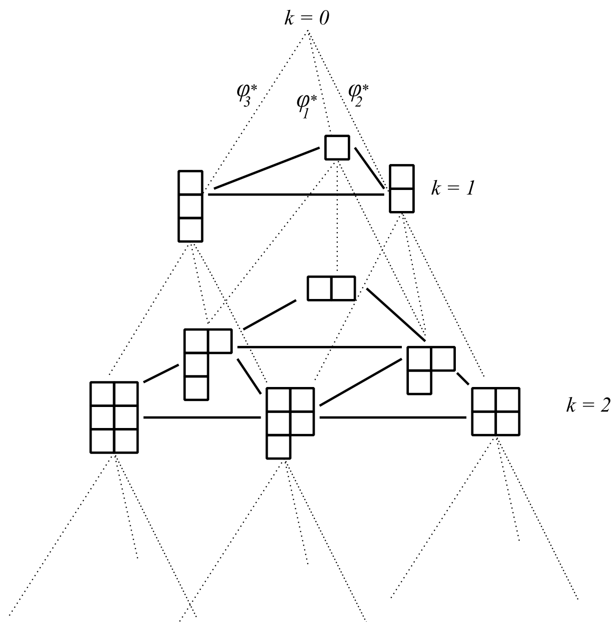


Figure 2.2: The tower of crystal graphs generated by the phase algebra for $n = 3$ and $k = 0, 1, 2$. The crystals for $k > 1$ all consist of triangles which are subdivided into smaller triangles similar as depicted for $k = 2$. The dotted lines correspond to the action of the phase algebra generators φ_i^* and the solid lines to the action of the plactic algebra $\mathrm{Pl}(\mathcal{A})$.

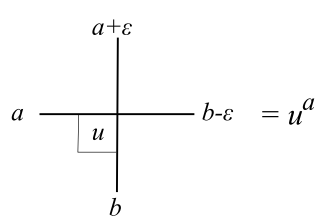


Figure 3.1: Graphical depiction of a vertex configuration with Boltzmann weight (3.1). The statistical variables are constrained by $a, b, \varepsilon = d - a, c = b - \varepsilon \in \mathbb{Z}_{\geq 0}$.

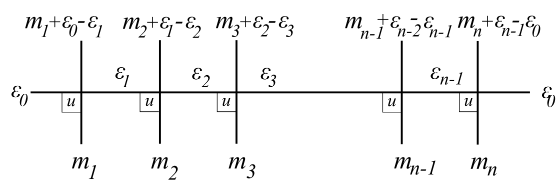


Figure 3.2: The allowed row configurations of the vertex model (3.1) with $\varepsilon_i, m_i, m_i - \varepsilon_i \in \mathbb{Z}_{\geq 0}$. Due to the periodic boundary conditions $\varepsilon_n = \varepsilon_0$.

The variable u is called the spectral parameter and we define $\mathcal{M}(u) = \mathbb{C}[u, u^{-1}] \otimes_{\mathbb{C}} \mathcal{M}$. The Boltzmann weights define an operator $\mathcal{R}(u/v) : \mathcal{M}(u) \otimes \mathcal{M}(v) \to \mathcal{M}(u) \otimes \mathcal{M}(v)$ via the relation

$$\mathcal{R}(u)\ v_a \otimes v_b = \sum_{c,d \geq 0} \mathcal{R}^{a,b}_{c,d}(u)\ v_c \otimes v_d, \tag{3.2}$$

which we can express in terms of the phase algebra generators (2.8): let $\mathcal{P}(v_a \otimes v_b) = v_b \otimes v_a$ be the flip operator then

$$\mathcal{R}(u) = \mathcal{P} \left[\sum_{\alpha \in \mathbb{Z}_{\geq 0}} (\varphi^*)^\alpha \otimes \varphi^\alpha \right] (u^N \otimes 1)\ . \tag{3.3}$$

Despite the infinite sum this operator is well-defined, since when acting on an arbitrary vector $v_a \otimes v_b \in \mathcal{M}(u) \otimes \mathcal{M}(v)$ only a finite number of terms in the sum are nonzero.

Following the standard procedure [1] we now employ the $\mathcal{R}$-matrix to define the discrete evolution operator, the row-to-row transfer matrix, of our statistical mechanics model. Its matrix elements are obtained by fixing two sets $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{H} = \mathcal{M}^{\otimes n}$ of statistical variables along the incoming and outgoing vertical edges of a lattice row and summing over those variables which sit at the horizontal edges; see Figure 3.2 for an allowed row configuration. As an operator the transfer matrix is given by

$$Q(u) = \operatorname*{Tr}_0 \left[z^{N \otimes 1} \mathcal{R}_{0n}(u) \cdots \mathcal{R}_{01}(u) \right] \in \operatorname{End} \mathcal{H}[u, z], \tag{3.4}$$

where $\mathcal{H}[u, z] := \mathbb{C}(u, z) \otimes_{\mathbb{C}} \mathcal{H}$ and z is an indeterminate (as we will see below the same as the one in Proposition 2.6). The lower indices $i = 1, \ldots, n$ refer to the n vertical edges in one row (the n factors of $\mathcal{M}$ in $\mathcal{H}$) and the index 0 belongs to the horizontal edges over which the sum is taken (the copy of $\mathcal{M}$ over which the trace is computed). The trace together with the additional operator $z^{N \otimes 1}$ enforces quasi-periodic boundary conditions in the horizontal lattice direction.

While the trace is taken over an infinite-dimensional vector space, $\mathcal{M}$, for any pair of configurations $\boldsymbol{m}, \boldsymbol{m}' \in \mathcal{H} = \mathcal{M}^{\otimes n}$ only a finite number of the terms making up the matrix element $\langle \boldsymbol{m}, Q(u) \boldsymbol{m}' \rangle$ is non-zero. The operator Q is therefore well-defined. We will show this in the proof of Proposition 4.1 below

by explicitly computing the matrix elements and showing that $Q(u)$ should be understood as formal power series in u with operator valued coefficients. First we wish to show integrability of our vertex model, i.e. that the operator Q commutes with itself for any pair of spectral parameters.

Proposition 3.1 *Let $\mathcal{R}$ be the operator (3.3) and define $\mathcal{S}(u/v) \in \mathrm{End}[\mathcal{M}(u) \otimes \mathcal{M}(v)]$ by setting*

$$\mathcal{S}(u) = (1-u)\mathcal{R}(u) + \mathcal{P}(u^{N+1} \otimes 1) \; . \tag{3.5}$$

Then we have the identity

$$\mathcal{S}_{12}(u)\mathcal{R}_{13}(uv)\mathcal{R}_{23}(v) = \mathcal{R}_{23}(v)\mathcal{R}_{13}(uv)\mathcal{S}_{12}(u), \tag{3.6}$$

where the lower indices indicate in which factor of the tensor product $\mathcal{M}(u) \otimes \mathcal{M}(uv) \otimes \mathcal{M}(v)$ the respective operators act non-trivially. Moreover, $\mathcal{S}$ is invertible, $\mathcal{S}^{-1}(u) = \mathcal{P}\mathcal{S}(u^{-1})\mathcal{P}$.

Proof. Because of the explicit appearance of the flip operator $\mathcal{P}$ in $\mathcal{R}, \mathcal{S}$ it is convenient to work with $\hat{\mathcal{R}} = \mathcal{P}\mathcal{R}$ and $\hat{\mathcal{S}} = \mathcal{P}\hat{\mathcal{S}}$. Then the Yang-Baxter equation is rewritten as follows

$$[1 \otimes \hat{\mathcal{S}}(u)][\hat{\mathcal{R}}(uv) \otimes 1][1 \otimes \hat{\mathcal{R}}(v)] = [\hat{\mathcal{R}}(v) \otimes 1][1 \otimes \hat{\mathcal{R}}(uv)][\hat{\mathcal{S}}(u) \otimes 1] \; .$$

We now compute the corresponding identity in terms of matrix elements by evaluating the identity on the vector $|a,b,c\rangle \equiv v_a \otimes v_b \otimes v_c$ and then multiplying from the left with the dual vector $\langle d,e,f| = v_d^* \otimes v_e^* \otimes v_f^*$. Both sides of the identity vanish (and it therefore holds trivially true) unless $c-f,\, d-a \geq 0$ and $a+b+c = d+e+f$ according to (3.1). Provided these conditions are satisfied, the computation of the left hand side yields,

$$\langle d,e,f|[1 \otimes \hat{\mathcal{S}}(u)][\hat{\mathcal{R}}(uv) \otimes 1][1 \otimes \hat{\mathcal{R}}(v)]|a,b,c\rangle = \\ (uv)^{a+b}(1-u) \sum_{i=\max(0,d-a-b)}^{c-f} u^{i-d+a} + (uv)^{a+b}u^{1+e-b},$$

while from the right hand side we obtain

$$\langle d,e,f|[\hat{\mathcal{R}}(v) \otimes 1][1 \otimes \hat{\mathcal{R}}(uv)][\hat{\mathcal{S}}(u) \otimes 1]|a,b,c\rangle = \\ (uv)^{a+b}(1-u) \sum_{i=0}^{\min(b,d-a)} u^{-i} + (uv)^{a+b}u \; .$$

Distinguishing the two cases $b \geq d-a$ and $b < d-a$ we now verify that the identity is true.

The formula for the inverse is verified in a similar manner, by acting with $\mathcal{S}$ on the subspace spanned by the vectors $v_a \otimes v_b$ with $a+b$ fixed. This subspace is finite-dimensional and the above formula can be explicitly computed. ■

Corollary 3.2 (Integrability) *The transfer matrix Q of our vertex model commutes for any pair (u,v) of spectral parameters, $[Q(u),Q(v)]=0$, and, hence, the model is integrable.*

Proof. After writing out the product $Q(u)Q(v)$ as a trace over $\mathcal{R}$-matrices according to the definition (3.4) and inserting the identity $1=\mathcal{S}(u)\mathcal{S}^{-1}(u)$ under the trace the assertion follows from the Yang-Baxter equation (3.6). ■

3.2 Relation with the phase model

To conclude this section we explain how the vertex model (3.1) is related to the phase model of Bogoliubov, Izergin and Kitanine [3]. The phase model was used in [16, Part I] to give a combinatorial description of the $\widehat{\mathfrak{sl}}(n)_k$ WZNW fusion ring. As we will show in the next sections we arrive at the same description of the fusion ring via the transfer matrix (3.4) which we obtained from the $U_q\widehat{\mathfrak{sl}}(2)$-vertex model in the crystal limit. We are therefore lead to investigate the relation between our vertex model and the phase model. We recall that the L-operator of the phase model is given by (cf. [3], [16, Sections 3.2 and 4])

$$L(u)=\sigma^+\sigma^-\otimes 1+\sigma^+\otimes\varphi+u\;\sigma^-\otimes\varphi^*+u\;\sigma^-\sigma^+\otimes 1\;, \tag{3.7}$$

where $\sigma^+=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\sigma^-=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$ are the Pauli matrices acting in $\mathbb{C}(u)^2$. The transfer matrix for the phase model then reads in analogy with (3.4),

$$T(u)=\operatorname*{Tr}_0 z^{\sigma^3\otimes 1}L_{0n}(u)\cdots L_{01}(u)\in\operatorname{End}\mathcal{H}[u,z], \tag{3.8}$$

where $\sigma^3=\sigma^-\sigma^+$ and the so-called auxiliary space indexed by "0", over which the trace is taken, is now $\mathbb{C}^2(u)$. The following proposition is obtained from a straightforward computation.

Proposition 3.3 *Define the following element in* $\operatorname{End}(\mathbb{C}(u)^2\otimes\mathcal{M})$*,*

$$L'(u)=L(u)+u\;\sigma^+\sigma^-\otimes(1-\varphi^*\varphi), \tag{3.9}$$

then we have in $\operatorname{End}[\mathbb{C}(u)^2\otimes\mathcal{M}(v)\otimes\mathcal{M}]$ *the identity*

$$L'_{12}(u/v)L_{13}(u)\mathcal{R}_{23}(v)=\mathcal{R}_{23}(v)L_{13}(u)L'_{12}(u/v)\;. \tag{3.10}$$

Remark 3.4 *Note that the operator L' does not possess an inverse. Similarly, like the other operators (3.3) and (3.5) it is another special crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$-intertwiner associated with $M_\mu(u)\otimes M_\nu(v)$; see the appendix.*

Within the context of the quantum inverse scattering method [4] one introduces the Yang-Baxter algebra which for the phase model is generated by the following matrix elements

$$\begin{pmatrix} A(u) & B(u)\\ C(u) & D(u)\end{pmatrix}:=\langle v^*_{\sigma'}\otimes 1, z^{\sigma^3\otimes 1}L_{0n}(u)\cdots L_{01}(u)v_\sigma\otimes 1\rangle_{\sigma',\sigma=0,1}\;. \tag{3.11}$$

The matrix elements $A, B, C, D \in \mathrm{End}\,\mathcal{H}[u,z]$ have a particularly simple combinatorial action; cf. [16, Cor 3.9]. Note that $T(u) = A(u) + zD(u)$. In analogy we now define for our vertex model (3.4) the infinite-dimensional operator-valued matrix

$$Q_{\varepsilon',\varepsilon}(u) := \langle v^*_{\varepsilon'} \otimes 1, z^{N\otimes 1}\mathcal{R}_{0n}(u)\cdots\mathcal{R}_{01}(u) v_\varepsilon \otimes 1\rangle \in \mathrm{End}\,\mathcal{H}[u,z] \tag{3.12}$$

with $\varepsilon, \varepsilon' \in \mathbb{Z}_{\geq 0}$ and $Q(u) = \sum_{\varepsilon\geq 0} Q_{\varepsilon,\varepsilon}(u)$. We compute these matrix elements explicitly in the next section and show that also they have a nice combinatorial interpretation. First we have the following consequence from the previous proposition.

Corollary 3.5 *The generators of the Yang-Baxter algebra for the phase model (3.8) and the vertex model (3.4) obey the following commutation relations:*

$$\begin{aligned}
A(u)Q_{\varepsilon',\varepsilon}(v) - Q_{\varepsilon',\varepsilon}(v)A(u) &= Q_{\varepsilon',\varepsilon-1}(v)B(u) - u/v\ C(u)Q_{\varepsilon'-1,\varepsilon}(v) \\
&\quad + \frac{u}{v}\left[\delta_{\varepsilon,0}Q_{\varepsilon',0}(v)A(u) - \delta_{\varepsilon',0}A(u)Q_{0,\varepsilon}(v)\right]
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
Q_{\varepsilon',\varepsilon}(v)B(u) - v/u\ B(u)Q_{\varepsilon',\varepsilon}(v) &= D(u)Q_{\varepsilon'-1,\varepsilon}(v) - Q_{\varepsilon',\varepsilon+1}(v)A(u) \\
&\quad + \delta_{\varepsilon',0}B(u)Q_{0,\varepsilon}(v)
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
Q_{\varepsilon',\varepsilon}(v)C(u) - u/v\ C(u)Q_{\varepsilon',\varepsilon}(v) &= A(u)Q_{\varepsilon'+1,\varepsilon}(v) - Q_{\varepsilon',\varepsilon-1}(v)D(u) \\
&\quad - \delta_{\varepsilon,0}\frac{u}{v}Q_{\varepsilon',0}(v)C(u)
\end{aligned} \tag{3.15}$$

$$D(u)Q_{\varepsilon',\varepsilon}(v) - Q_{\varepsilon',\varepsilon}(v)D(u) = Q_{\varepsilon',\varepsilon+1}(v)C(u) - v/u\ B(u)Q_{\varepsilon'+1,\varepsilon}(v) \tag{3.16}$$

Here matrix elements with negative indices are understood to be zero. In particular, we have

$$D(u)Q_{\varepsilon',\varepsilon}(v) - Q_{\varepsilon',\varepsilon}(v)D(u) = Q_{\varepsilon'+1,\varepsilon+1}(v)A(u) - A(u)Q_{\varepsilon'+1,\varepsilon+1}(v)\ . \tag{3.17}$$

Proof. Set $\mathcal{T}(u) = z^{\sigma^3\otimes 1}L_{0n}(u)\cdots L_{01}(u)$ and $\mathcal{Q}(v) = z^{N\otimes 1}\mathcal{R}_{0n}(v)\cdots\mathcal{R}_{01}(v)$. Then the first four commutation relations are easily obtained by considering the following equality of matrix elements

$$\begin{aligned}
&\langle v^*_{\sigma'} \otimes v^*_{\varepsilon'} \otimes 1, L'_{12}(u/v)\mathcal{T}_{13}(u)\mathcal{Q}_{23}(v) v^*_\sigma \otimes v_\varepsilon \otimes 1\rangle = \\
&\qquad\qquad \langle v^*_{\sigma'} \otimes v^*_{\varepsilon'} \otimes 1, \mathcal{Q}_{23}(v)\mathcal{T}_{13}(u)L'_{12}(u/v) v^*_\sigma \otimes v_\varepsilon \otimes 1\rangle
\end{aligned}$$

which follows from (3.10). For the last identity (3.17) employ the equality (3.16) together with the relations

$$C(u) = \varphi_n A(u) \qquad \text{and} \qquad B(u) = uA(u)\varphi_1^*, \tag{3.18}$$

which follow from the definition (3.7) and (3.11). ■

4 Noncommutative symmetric polynomials

We now connect the definition of our infinite-dimensional vertex model in terms of the transfer matrix (3.4) with the discussion of the plactic algebra in Section 2.2. We show that the matrix elements of Q can be written as analogues

of the complete symmetric functions in a noncommutative alphabet, the local affine plactic algebra $\mathrm{Pl}(\mathcal{A})$ in the representation (2.18). Recall that given a set of *commutative* indeterminates $x = (x_1, \ldots, x_n)$, the complete symmetric functions are the coefficients in the formal power series expansion of the following generating function [18, Chapter I, Section 2, p21],

$$H(u) = \prod_{i=1}^{n} \frac{1}{1 - x_i u} = \sum_{r \geq 0} h_r(x_1, \ldots, x_n) u^r \ . \tag{4.1}$$

This definition implies the following recursive formula with respect to n,

$$h_r(x_1, \ldots, x_n) = h_r(x_1, \ldots, x_{n-1}) + x_n h_{r-1}(x_1, \ldots, x_n) \ . \tag{4.2}$$

The solution is given by the following explicit expression

$$h_r(x_1, \ldots, x_n) = \sum_{p \vdash r} x_1^{p_1} \cdots x_n^{p_n}, \tag{4.3}$$

where the sum runs over all compositions p of $r > 0$ and $h_0 = 1$. Up to a specific ordering we now show that the same formulae hold for a series expansion of (3.4) when replacing the commutative variables $(x_1, \ldots, x_n)$ with the generators of $\mathrm{Pl}(\mathcal{A})$, i.e. the Q-operator is the generating function complete symmetric polynomials in a noncommutative alphabet.

Proposition 4.1 *Let $Q(u)$ be the operator defined in (3.4) and denote by φ_i, φ_i^* and $a_i = \varphi_i \varphi_{i+1}^*$ the generators of the phase and local affine plactic algebra; see Sections 2.1 and 2.2. Then we have the following formal power series expansion*

$$Q(u) = \operatorname*{Tr}_0 z^{N \otimes 1} \mathcal{R}_{0n}(u) \cdots \mathcal{R}_{01}(u) = \sum_{r \geq 0} u^r h_r(\mathcal{A}), \tag{4.4}$$

where

$$h_r(\mathcal{A}) := \sum_{|\boldsymbol{\varepsilon}| = r} z^{\varepsilon_0} (\varphi_1^*)^{\varepsilon_0} a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_{n-1}^{\varepsilon_{n-1}} \varphi_n^{\varepsilon_0} \tag{4.5}$$

with the sum running over all compositions $\boldsymbol{\varepsilon} = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1})$ and $|\boldsymbol{\varepsilon}| = \sum_i \varepsilon_i$. In particular, when setting $z = 0$ we obtain that $Q_{0,0}(u) = \sum_{r \geq 0} u^r h_r(\mathcal{A}')$, where $h_r(\mathcal{A}') = \sum_{|\boldsymbol{\varepsilon}| = r, \varepsilon_0 = 0} a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_{n-1}^{\varepsilon_{n-1}}$ are the complete symmetric polynomials in the finite *plactic algebra $\mathrm{Pl}_{\mathrm{fin}}(\mathcal{A}')$ and analogous to (4.2) we have the recursion relation*

$$h_r(\mathcal{A}) = h_r(\mathcal{A}') + z \varphi_1^* h_{r-1}(\mathcal{A}) \varphi_n \ . \tag{4.6}$$

Remark 4.2 *To facilitate the comparison with the commutative case, assume that there exists one summand in the sum (4.5) for which ε_j vanishes. Then the corresponding monomial can be rewritten as ($\varepsilon_j = 0$)*

$$\begin{aligned} z^{\varepsilon_0} (\varphi_1^*)^{\varepsilon_0} a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_{n-1}^{\varepsilon_{n-1}} \varphi_n^{\varepsilon_0} = \\ z^{\varepsilon_0} a_{j+1}^{\varepsilon_{j+1}} a_{j+2}^{\varepsilon_{j+2}} \cdots a_{n-1}^{\varepsilon_{n-1}} \varphi_n^{\varepsilon_0} (\varphi_1^*)^{\varepsilon_0} a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_{j-1}^{\varepsilon_{j-1}} = \\ a_{j+1}^{\varepsilon_{j+1}} a_{j+2}^{\varepsilon_{j+2}} \cdots a_{n-1}^{\varepsilon_{n-1}} a_0^{\varepsilon_0} a_1^{\varepsilon_1} a_2^{\varepsilon_2} \cdots a_{j-1}^{\varepsilon_{j-1}}, \end{aligned}$$

where we have exploited that $[a_i,a_j]=0$ for $|i-j|\bmod n>1$ and $a_n=z\varphi_n\varphi_1^*$. In particular, if $r<n$ then we can always find for each summand in (4.5) some $1\leq j\leq n$ such that $\varepsilon_j=0$ and the definition of the complete symmetric polynomials coincides with the one in [16, Def 5.16],

$$r<n:\ h_r(\mathcal{A})=\sum_{p\vdash r}\prod_{i=1}^{\circlearrowright}a_i^{p_i},\qquad \prod_{i=1}^{\circlearrowright}a_i^{p_i}=a_{j+1}^{\varepsilon_{j+1}}\cdots a_{n-1}^{\varepsilon_{n-1}}a_0^{\varepsilon_0}a_1^{\varepsilon_1}\cdots a_{j-1}^{\varepsilon_{j-1}},$$

where the letters are clockwise cyclically ordered. The similarity with (4.3) is now apparent. The new result here, compared to [16, Cor 6.9], is the explicit expression for $h_r(\mathcal{A})$ when $r\geq n$.

Proof. The proof is immediate from the definition of the Boltzmann weights and the action of the phase algebra. Namely, consider an allowed row configuration as depicted in Figure 3.2. Summing over the variables located at the horizontal edges $\boldsymbol{\varepsilon}=(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1})$ we obtain the matrix element

$$\begin{aligned}\langle\boldsymbol{m}'|Q(u)|\boldsymbol{m}\rangle&=\sum_{\varepsilon_0}\langle\boldsymbol{m}'|Q_{\varepsilon_0,\varepsilon_0}(u)|\boldsymbol{m}\rangle\\&=\sum_{\boldsymbol{\varepsilon}}z^{\varepsilon_0}u^{\varepsilon_0+\cdots+\varepsilon_{n-1}}\langle\boldsymbol{m}'|(\varphi_1^*)^{\varepsilon_0}\varphi_1^{\varepsilon_1}(\varphi_2^*)^{\varepsilon_1}\varphi_2^{\varepsilon_2}\cdots(\varphi_n^*)^{\varepsilon_{n-1}}\varphi_n^{\varepsilon_0}|\boldsymbol{m}\rangle,\end{aligned}$$

where $|\boldsymbol{m}|=|\boldsymbol{m}'|=k\geq 0$. If $|\boldsymbol{m}|\neq|\boldsymbol{m}'|$ the matrix element vanishes according to (3.1). Note in particular that because of the ordering of the phase algebra generators it follows for $r>k$ that $h_r(\mathcal{A})\mathcal{H}_k=\{0\}$ and, thus, the series expansion terminates after finitely many summands on each $\mathcal{H}_k$. The operator (3.4) is therefore well defined as claimed earlier. The result (4.4) with (4.5) now follows from (2.18). ■

Remark 4.3 *As shown in [16, Prop 5.13] the transfer matrix (3.8) of the phase model is the generating function for the noncommutative analogues of the elementary symmetric polynomials,*

$$T(u)=\sum_{r=0}^{n}u^re_r(\mathcal{A}),\qquad e_r(\mathcal{A})=\sum_{\substack{p\vdash r\\p_i=0,1}}\prod_{i=1}^{\circlearrowleft}a_i^{p_i}\tag{4.7}$$

where the letters are now anticlockwise cyclically ordered and $e_n(\mathcal{A})=z\cdot 1$. Also in this instance the familiar recursion relation from the commutative case, $e_r(x_1,\ldots,x_n)=e_r(x_1,\ldots,x_{n-1})+x_ne_{r-1}(x_1,\ldots,x_{n-1})$ with generating function $E(u)=\prod_{i=1}^n(1+ux_i)=\sum_{r\geq 0}u^re_r(x_1,\ldots,x_n)$, generalises to the noncommutative case,

$$e_r(\mathcal{A})=e_r(\mathcal{A}')+z\varphi_ne_{r-1}(\mathcal{A}')\varphi_1^*\,.\tag{4.8}$$

This last equality is implicit in the results of [16, Prop 5.13]. Setting $z=0$ one infers that

$$A(u)=(1+ua_{n-1})(1+ua_{n-2})\cdots(1+ua_1)=\sum_{r\geq 0}u^re_r(\mathcal{A}')$$

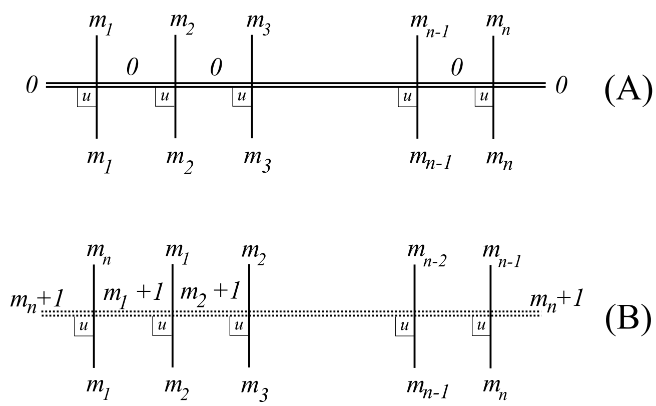


Figure 4.1: The allowed row configurations in the decomposition of the product $T(-u)Q(u)$; see the proof of Proposition 4.5. Here the double solid lines stand for the subspace W and the double dashed lines for the complement $\bar{W}$.

and the identity (4.8) then follows from (3.18).

Corollary 4.4 *The elementary and complete symmetric polynomials (4.5) in the noncommutative alphabet $\mathcal{A}$ pairwise commute,*

$$[e_r(\mathcal{A}), e_{r'}(\mathcal{A})] \overset{(1)}{=} [h_r(\mathcal{A}), h_{r'}(\mathcal{A})] \overset{(2)}{=} [e_r(\mathcal{A}), h_{r'}(\mathcal{A})] \overset{(3)}{=} 0\;. \tag{4.9}$$

Proof. The first equality is a result of [16, Cor 5.14]. The second equality in (4.9) is a direct consequence of Corollary 3.2. Finally, to prove the third equality we employ (3.17) to arrive at

$$T(u)Q(v) - Q(v)T(u) = Q_{0,0}(v)A(u) - A(u)Q_{0,0}(v) = 0\;. \tag{4.10}$$

Thus, $[e_r(\mathcal{A}), h_{r'}(\mathcal{A})] = -[e_r(\mathcal{A}'), h_{r'}(\mathcal{A}')] = 0$. That the last commutator in the finite plactic algebra vanishes is a direct consequence of (3.13) for $\varepsilon = \varepsilon' = 0$. ■

Proposition 4.5 (TQ-equation) *Let T and Q be the transfer matrices (3.8) and (3.4), respectively. Then they satisfy the following identity*

$$\begin{aligned} T(-u)Q(u) &= \left[Q(uq) + z(-u)^n q^K Q(uq^{-1})\right]_{q=0} \\ &= 1 + z(-u)^n \sum_{k\geq 0} u^k h_k(\mathcal{A})\pi_k\;, \end{aligned} \tag{4.11}$$

where $K = \sum_i N_i$ and π_k is the (orthogonal) projector onto $\mathcal{H}_k \subset \mathcal{H}$.

Proof. Let L' be the operator defined in (3.9). Then $W = \ker L'(-1) \subset \mathbb{C}^2 \otimes \mathcal{M}$ and the complement $\bar{W} \cong (\mathbb{C}^2 \otimes \mathcal{M})/W$ are spanned by the vectors

$$w_m := \begin{cases} v_0 \otimes v_0, & m = 0 \\ v_0 \otimes v_m + v_1 \otimes v_{m-1}, & m > 0 \end{cases} \qquad \text{and} \qquad \bar{w}_m := v_0 \otimes v_{m+1}\;,$$

respectively. From (3.10) we infer that $L_{13}(-u)\mathcal{R}_{23}(u)W\otimes\mathcal{M}\subset W\otimes\mathcal{M}$. In fact, we have that

$$L_{13}(-u)\mathcal{R}_{23}(u)w_m\otimes v_a=\delta_{m,0}\sum_{b=0}^{a}w_b\otimes v_{a-b} \tag{4.12}$$

and

$$L_{13}(-u)\mathcal{R}_{23}(u)\bar{w}_m\otimes v_a=-u^m\bar{w}_a\otimes v_{m-1}+\ \dots\ , \tag{4.13}$$

where the omitted terms in the second equality lie in W. Thus, we may write

$$\begin{aligned}T(-u)Q(u)=\operatorname*{Tr}_{\mathbb{C}^2\otimes\mathcal{M}}[z^{\sigma^3\otimes N\otimes 1}L_{0n}(-u)\mathcal{R}_{0'n}(u)\cdots L_{01}(-u)\mathcal{R}_{0'1}(u)]=\\ \operatorname*{Tr}_{W}[z^{\sigma^3\otimes N\otimes 1}L_{0n}(-u)\mathcal{R}_{0'n}(u)\cdots L_{01}(-u)\mathcal{R}_{0'1}(u)]\\ +\operatorname*{Tr}_{\bar{W}}[z^{\sigma^3\otimes N\otimes 1}L_{0n}(-u)\mathcal{R}_{0'n}(u)\cdots L_{01}(-u)\mathcal{R}_{0'1}(u)],\end{aligned}$$

where the indices 0 and $0'$ refer to the factor $\mathbb{C}^2$ and $\mathcal{M}$ in $\mathbb{C}^2\otimes\mathcal{M}$, respectively. The assertion now follows by observing that (4.12) and (4.13) imply that the only allowed vertex configurations in a row are the ones depicted in Figure 4.1. The configuration labelled (A) corresponds to the trace over W and yields the identity in (4.11).

The second configuration (B) describes the trace over $\bar{W}$ and coincides with the action of the $\widehat{\mathfrak{sl}}(n)$-Dynkin diagram automorphism, rot : $\boldsymbol{m}\mapsto(m_n,m_1,\dots,m_{n-1})$, which for $z=1$ is identical with the action of $h_k(\mathcal{A})$; see (4.5). ■

The following result is contained in [16, Part I, Def 5.16 and Cor 6.9] for $r<n$. Here we state an alternative proof valid for all $r>0$.

Corollary 4.6 *The familiar determinant relations from the commutative case also hold for the noncommutative elementary and complete symmetric polynomials,*

$$h_r(\mathcal{A})=\det(e_{1-i+j}(\mathcal{A}))_{1\le i,j\le r},\qquad e_r(\mathcal{A})=\det(h_{1-i+j}(\mathcal{A}))_{1\le i,j\le r}\ , \tag{4.14}$$

where the determinants are well defined due to (4.9).

Proof. Performing a series expansion in (4.11) with respect to the spectral parameter u we find for $j=1,2,\dots,n$ the identities

$$\sum_{r=0}^{j}(-1)^r e_r(\mathcal{A})h_{j-r}(\mathcal{A})=0\ , \tag{4.15}$$

which constitute a system of homogeneous linear equations in a set of commutative variables due to (4.9). The solution is therefore identical to the commutative case and is given by (4.14); see [18, page 21, eqn (2.6')]. The second term on the right hand side of equation (4.11) yields the equality

$$\sum_{r=0}^{n}(-1)^r e_r(\mathcal{A})h_{n+k-r}(\mathcal{A})\pi_k=(-1)^n z h_k(\mathcal{A})\pi_k$$

which is easily verified by observing that $e_n(\mathcal{A}) = z1$ and $h_r(\mathcal{A})\pi_k = 0$ for $r > k$ as discussed earlier. ■

5 The WZNW fusion ring

In this final section we explain how the ring of noncommutative functions generated from the transfer matrix (3.4) for the vertex model (3.1) and the transfer matrix (3.8) for the phase model is related to the fusion ring of $\widehat{\mathfrak{sl}}(n)_k$ WZNW conformal field theory. First we need to introduce special elements in the ring of noncommutative functions. We do this in a similar manner as in [16, Cor 6.8 and Cor 7.2] and therefore omit the proof.

Proposition 5.1 (Cauchy-identity) *Let λ be a partition and define the following noncommutative analogue of Schur polynomials*

$$s_\lambda(\mathcal{A}) = \det[h_{\lambda_i - i + j}(\mathcal{A})]_{1\le i,j\le n}\ . \tag{5.1}$$

In particular, we have $s_{(r)}(\mathcal{A}) = h_r(\mathcal{A})$ and $s_{(1^r)}(\mathcal{A}) = e_r(\mathcal{A})$, where (r) and (1^r) are a horizontal and vertical strip of length r. Then we have the generalised Cauchy identity

$$Q(u_1)\cdots Q(u_l) = \sum_\lambda s_\lambda(u_1,\dots,u_l)s_\lambda(\mathcal{A}) \tag{5.2}$$

for all $l > 0$. Here $s_\lambda(u_1,\dots,u_l)$ is the standard, commutative *Schur polynomial.*

Let us now recall the definition of the fusion ring $\mathcal{F}_k(\widehat{\mathfrak{sl}}(n),\mathbb{Z})$. The basic building blocks of the WZNW conformal field theory are the primary fields which can be viewed as the highest weight vectors with respect to the actions of both, the Virasoro algebra and the affine algebra $\widehat{\mathfrak{sl}}(n)$. Hence, as a set the primary fields are in one-to-one correspondence with certain elements in the weight lattice of $\widehat{\mathfrak{sl}}(n)$ which we now describe.

We identify the basis vectors of $\mathcal{H}_k$ in (2.15) labeled by P_k^+ (the finite set of partitions $\hat\lambda$ of maximal height n and of width $\hat\lambda_1 = k$) with the *integral dominant weights* of the affine algebra $\widehat{\mathfrak{sl}}(n)$ at level $k \in \mathbb{Z}_{\ge 0}$. Namely, given a partition $\hat\lambda$ we map to the weight $\sum_{i=1}^n m_i(\hat\lambda)\hat\omega_i$, where the coefficients $m_i(\hat\lambda)$ are the multiplicities of columns of height i and the $\hat\omega_i$ are the fundamental $\widehat{\mathfrak{sl}}(n)$ weights with $\hat\omega_n \equiv \hat\omega_0$; for details the reader is referred to [13] and [16, Part I, Section 2]. By abuse of notation we shall not distinguish between partitions and weights.

Given two primary fields associated with two $\widehat{\mathfrak{sl}}(n)$ weights at level k, say $\hat\lambda$ and $\hat\mu$, their fusion product can be expanded again into a sum of primary fields; see e.g. [5]. Thus, we now consider the free abelian group $\mathcal{F}_k(\widehat{\mathfrak{sl}}(n),\mathbb{Z})$ generated by the elements in P_k^+ with respect to addition and introduce for $\hat\lambda,\hat\mu \in P_k^+$ the

fusion product as follows

$$\hat{\lambda} * \hat{\mu} = \sum_{\hat{\nu} \in P_k^+} \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} \hat{\nu}, \qquad \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} = \sum_{\hat{\sigma} \in P_k^+} \frac{\mathcal{S}_{\hat{\lambda}\hat{\sigma}} \mathcal{S}_{\hat{\mu}\hat{\sigma}} \bar{\mathcal{S}}_{\hat{\nu}\hat{\sigma}}}{\mathcal{S}_{0\hat{\sigma}}} \,. \tag{5.3}$$

The structure constants $\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}}$, known as fusion coefficients in the physics literature, are given in terms of the Verlinde formula [21] with $\mathcal{S}_{\hat{\lambda}\hat{\mu}}$ denoting a matrix element of the modular $\mathcal{S}$-matrix which is explicitly given in terms of the Kac-Peterson formula [14] ($\iota = \sqrt{-1}$),

$$\mathcal{S}_{\hat{\lambda}\hat{\sigma}} = \frac{e^{\iota \pi n(n-1)/4}}{\sqrt{n(k+n)^{n-1}}} \sum_{w \in S_n} (-1)^{\ell(w)} e^{-\frac{2\pi\iota}{k+n}(\sigma+\rho, w(\lambda+\rho))} \tag{5.4}$$

Here $\lambda = \hat{\lambda} - k\hat{\omega}_0$ is the finite part of the affine weight and $\rho = \sum_{i=1}^{n-1} \omega_i$ is the Weyl vector with ω_i being the finite fundamental weights of $\mathfrak{sl}(n)$.

Theorem 5.2 (Korff, Stroppel [16]) *Let $\hat{\lambda}, \hat{\mu}, \hat{\nu} \in P_k^+$ and set $z = 1$. Then we have the identity*

$$\hat{\lambda} * \hat{\mu} = s_{\hat{\lambda}}(\mathcal{A})\hat{\mu} \tag{5.5}$$

and in particular the following equality holds, $\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} = \langle \hat{\nu}, s_{\hat{\lambda}}(\mathcal{A})\hat{\mu} \rangle$.

Remark 5.3 *Setting alternatively $z = 0$ we specialise to the ring of noncommutative Schur polynomials $s_\lambda(\mathcal{A}')$ in the local finite plactic algebra [7]. According to [16, Lemma 6.3 and Theorem 6.20] one then obtains the following quotient of the cohomology ring of the Grassmannian $\mathrm{Gr}_{k,n+k-1}$, $\mathrm{H}^*(\mathrm{Gr}_{\mathrm{k,n+k-1}})/\langle h_{n+k} \rangle \cong \mathbb{Z}[e_1, \ldots, e_k]/\langle h_n, \ldots, h_{n+k} \rangle$, whose structure constants are the intersection numbers $c_{\lambda\mu}^\nu$ of three hyperplanes with $\mu_1 = \nu_1$ and coincide with the celebrated Littlewood-Richardson coefficients [9, §9.4, Exercise 21 (a)].*

Proof. The proof of this result can be found in detail in [16, Part I, Section 6] and relies on the explicit construction of an eigenbasis for the transfer matrix (3.8) of the phase model (i.e. the generating function of the noncommutative elementary symmetric polynomials (4.7)) via the quantum inverse scattering method or algebraic Bethe Ansatz; see e.g. [4]. Because of the relations (4.14) and (5.1) this eigenbasis, called Bethe vectors, forms also an eigenbasis of the other noncommutative symmetric polynomials and in particular of the transfer matrix (3.4). One then verifies that the transformation matrix from the standard basis labeled by $\hat{\lambda} \in P_k^+$ to the Bethe vectors is given by the modular S-matrix (5.4). From this result one derives the Verlinde formula (5.3) for the matrix elements of the noncommutative Schur polynomial and, thus, the identity (5.5) follows. ■

We conclude by stating two corollaries which are now obvious consequences of the last Theorem. We therefore omit their proofs. The first one uses the recursion formula (4.6) for noncommutative complete symmetric polynomials to relate fusion coefficients at different level k; this is in analogy with the recursion relation in [16, Cor 7.4].

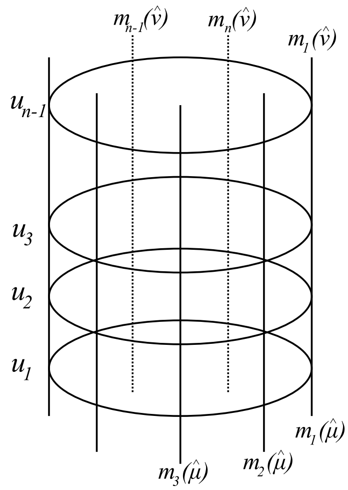


Figure 5.1: Graphical depiction of the $(n-1)\times n$ lattice with periodic boundary conditions in the horizontal direction and fixed boundary conditions $\hat{\mu},\hat{\nu}\in P_k^+$ on the outer vertical edges. The spectral parameter varies from row to row. The corresponding partition function obtained by summing over the Boltzmann weights (3.1) at each vertex yields (5.7).

Corollary 5.4 (Recursion relation) *For fixed level $k\in\mathbb{Z}_{\geq 0}$ let $\lambda=(r)$ be a horizontal strip of length $r\leq k$ and set $\hat{\lambda}_r=(k,k-r,\dots,k-r)\in P_k^+$. Then we have the following recursion relation for fusion coefficients,*

$$\mathcal{N}^{(k),\hat{\nu}}_{\hat{\lambda}_r,\hat{\mu}}=c^{\hat{\nu}}_{\hat{\mu}(r)}+\mathcal{N}^{(k-1),\varphi_1\hat{\nu}}_{\hat{\lambda}_{r-1},\varphi_n\hat{\mu}}\;, \tag{5.6}$$

where $c^{\nu}_{\lambda\mu}=c^{\nu}_{\mu\lambda}$ is the Littlewood-Richardson coefficient.

Note that while this relation only involves horizontal strips the latter allow one to compute all fusion coefficients via (5.1) and (5.5).

The second consequence of the above Theorem and the identity (5.2) is the interpretation of the partition function of the vertex model (3.1).

Corollary 5.5 (Generating function for fusion coefficients) *Given $\hat{\mu},\hat{\nu}\in P_k^+$ consider the vertex model (3.1) on an $(n-1)\times n$ lattice with periodic boundary conditions in the horizontal direction and fix the boundary conditions in the vertical directions by $\hat{\mu}$ and $\hat{\nu}$; see Figure 5.1. Assign to each row the spectral parameter u_i, then the corresponding partition function (i.e. the weighted sum over all allowed vertex configurations) has the expansion*

$$\begin{aligned} Z^{\hat{\nu}}_{\hat{\mu}}(u_1,\dots,u_{n-1}) &= \langle m(\hat{\nu})|Q(u_1)\cdots Q(u_{n-1})|m(\hat{\mu})\rangle \\ &= \sum_{\hat{\lambda}\in P_k^+}\mathcal{N}^{(k),\hat{\nu}}_{\hat{\lambda}\hat{\mu}}\,s_\lambda(u_1,\dots,u_{n-1}), \end{aligned} \tag{5.7}$$

where λ is the partition obtained from $\hat{\lambda}$ by deleting all columns of height n. Therefore we might interpret Z as generating function for the fusion coefficients.

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Appendix

A Derivation of the vertex model from $U_q\widehat{\mathfrak{sl}}(2)$

In this appendix we describe how the matrices (3.3) and (3.5) are obtained as a special limit from the $U_q\widehat{\mathfrak{sl}}(2)$-intertwiner $R(u,v;\mu,\nu):M_\mu(u)\otimes M_\nu(v)\to M_\mu(u)\otimes M_\nu(v)$ satisfying the relation

$$R\Delta(X)=\Delta^{\rm op}(X)R,\qquad X\in U_q\widehat{\mathfrak{sl}}(2)\;, \tag{A.1}$$

where $\Delta^{\rm op}$ is the coproduct (2.3) with the order of the factors interchanged. Given that the coproducts Δ and $\Delta^{\rm op}$ are algebra homomorphisms it is sufficient to solve the intertwining relation for $X=E_i,F_i,K_i^{\pm 1}$. These identities provide us with a set of equations for the matrix elements of $R=R(u,v;\mu,\nu)$ which enable us to compute them recursively. For convenience we make the parameter transformations $\mu\to q/s$ and $v\to s^{-1}$. Setting as before $R(v_a\otimes v_b)=\sum_{c,d\geq 0}R^{a,b}_{c,d}\;v_c\otimes v_d$ one obtains for $X=K_i^{\pm1}$ the constraint

$$R^{a,b}_{c,d}=0\quad\text{unless}\quad a+b=c+d, \tag{A.2}$$

and for $X=E_0,F_1$ the recursion relations

$$R^{a+1,b}_{c,d}\;=\;\frac{(q^{2d+1}-us^2\nu q^{2a})R^{a,b}_{c-1,d}+us\nu(1-s^2q^{2(a+c)})R^{a,b}_{c,d-1}}{\nu-us^2q^{2(a+b)+1}}\;, \tag{A.3}$$

$$R^{a,b+1}_{c,d}\;=\;\frac{(\nu-\nu^{-1}q^{2(d+b+1)})R^{a,b}_{c-1,d}+s(\nu q^{2c}-uq^{2b+1})R^{a,b}_{c,d-1}}{\nu-us^2q^{2(a+b)+1}}\;, \tag{A.4}$$

where matrix elements with negative indices are understood to be zero. Note that as long as u,μ,ν are generic these homogeneous relations determine R up to a scalar factor. We choose the convention $R^{0,0}_{0,0}=1$, then the above relations allow us to successively compute all other matrix elements. It follows from the general axioms of a quasi-triangular Hopf algebra (see e.g. [6, Section 4.2, Prop 4.2.7]) that the result yields a solution to the Yang-Baxter equation.

Since $R^{0,0}_{0,0}=1$ and all the coefficients in the recursion relations are regular at $q=0$ we can conclude that $R^{a,b}_{c,d}\in\mathbb{A}$. Let $\mathbb{J}\subset\mathbb{A}$ be the ideal generated by q

and denote by $\tilde{R}^{a,b}_{c,d}$ the image of $R^{a,b}_{c,d}$ under the isomorphism $\mathbb{A}/\mathbb{J} \to \mathbb{C}$ then

$$\tilde{R}^{a+1,b}_{c,d} = -us^2\delta_{a,0}\tilde{R}^{0,b}_{c-1,d} + us(1-s^2\delta_{a,0}\delta_{c,0})\tilde{R}^{0,b}_{0,d-1}\ , \tag{A.5}$$

$$\tilde{R}^{a,b+1}_{c,d} = \tilde{R}^{a,b}_{c-1,d} + s\delta_{c,0}\tilde{R}^{a,b}_{0,d-1}\ , \tag{A.6}$$

Note that these relations are now independent of the parameter $\nu \in \mathbb{C}$. The solution to these equations can be explicitly written down, the non-vanishing matrix elements are

$$\tilde{R}^{0,b}_{c,b-c} = s^{b-c}, \qquad \tilde{R}^{a,b}_{b+1,a-1} = -u^a s^{a+1}, \tag{A.7}$$

$$\tilde{R}^{a,b}_{b-\varepsilon,a+\varepsilon} = u^a s^{a+\varepsilon}(1-s^2) \quad \text{for} \quad a>0,\ 0 \le \varepsilon \le b\ . \tag{A.8}$$

To simplify the result further we now take the limit $\mathcal{R}^{a,b}_{c,d}(u) = \lim_{s\to 0} s^{-d}\tilde{R}^{a,b}_{c,d}$ and obtain (3.1).

The derivation of the operator (3.5) follows along similar lines. The $U_q\widehat{\mathfrak{sl}}(2)$-intertwiner must satisfy the following identity in $\mathrm{End}[M_\mu(u)\otimes M_\mu(v)\otimes M_\nu(1)]$,

$$R_{12}(u/v;\mu,\mu)R_{13}(u;\mu,\nu)R_{23}(v;\mu,\nu) = R_{23}(v;\mu,\nu)R_{13}(u;\mu,\nu)R_{12}(u/v;\mu,\mu)\ .$$

Once more, the Yang-Baxter equation is a direct consequence of one of the axioms of a quasi-triangular Hopf algebra such as $U_q\widehat{\mathfrak{sl}}(2)$. Notice the different dependence on the parameters μ and ν in the equation. Similar as before we can consider the value $\tilde{R}$ of the matrix elements $R^{a,b}_{c,d}(us/v;q/s,q/s)$ at $q=0$ and then take the limit $\mathcal{S}(u)^{a,b}_{c,d} := \lim_{s\to 0}\ s^{c-b}\tilde{R}^{a,b}_{c,d}$ to find the operator (3.5).

It turns out that also the phase model is obtained from the $U_q\widehat{\mathfrak{sl}}(2)$-intertwiner for $M_\mu(u)\otimes M_\nu(v)$ in the crystal limit, albeit choosing a different specialisation for μ. Namely, set $\mu = q^2$ then the crystal limit of $(1\otimes q^N)R(-uq;q^2,\nu)$ is the operator (3.7) over $\mathbb{C}(u)^2\otimes\mathcal{M}\subset M_{\mu=q^2}(u)\otimes\mathcal{M}$, where we identify the two-dimensional subspace in $M_{\mu=q^2}(u)$ spanned by $\{v_0,v_1\}$ with $\mathbb{C}(u)^2$. This reduction to a subspace is justified by observing that for $\mu = q^2$ it can be mapped onto the fundamental evaluation module of $U_q\widehat{\mathfrak{sl}}(2)$; see Remark 2.2.

Finally, also the operator L' is another special crystal limit of the $U_q\widehat{\mathfrak{sl}}(2)$-intertwiner determined by (A.2) and the recursion relations (A.3) and (A.4). We summarise the various relations in the following table (as before we set $s\to 0$ after taking the crystal limit):

$U_q\widehat{\mathfrak{sl}}(2)$-intertwiner	crystal limit	μ	ν
$(1\otimes s^{-N})R(us)$	$\mathcal{R}(u)$	q/s	arbitrary
$(s^N\otimes 1)R(us)(1\otimes s^{-N})$	$\mathcal{S}(u)$	q/s	q/s
$(1\otimes q^N)R(-uq)$	$L(u)$	q^2	arbitrary
$(1\otimes q^N)R(-uq/s)(s^N\otimes 1)$	$L'(u)$	q^2	q/s

.

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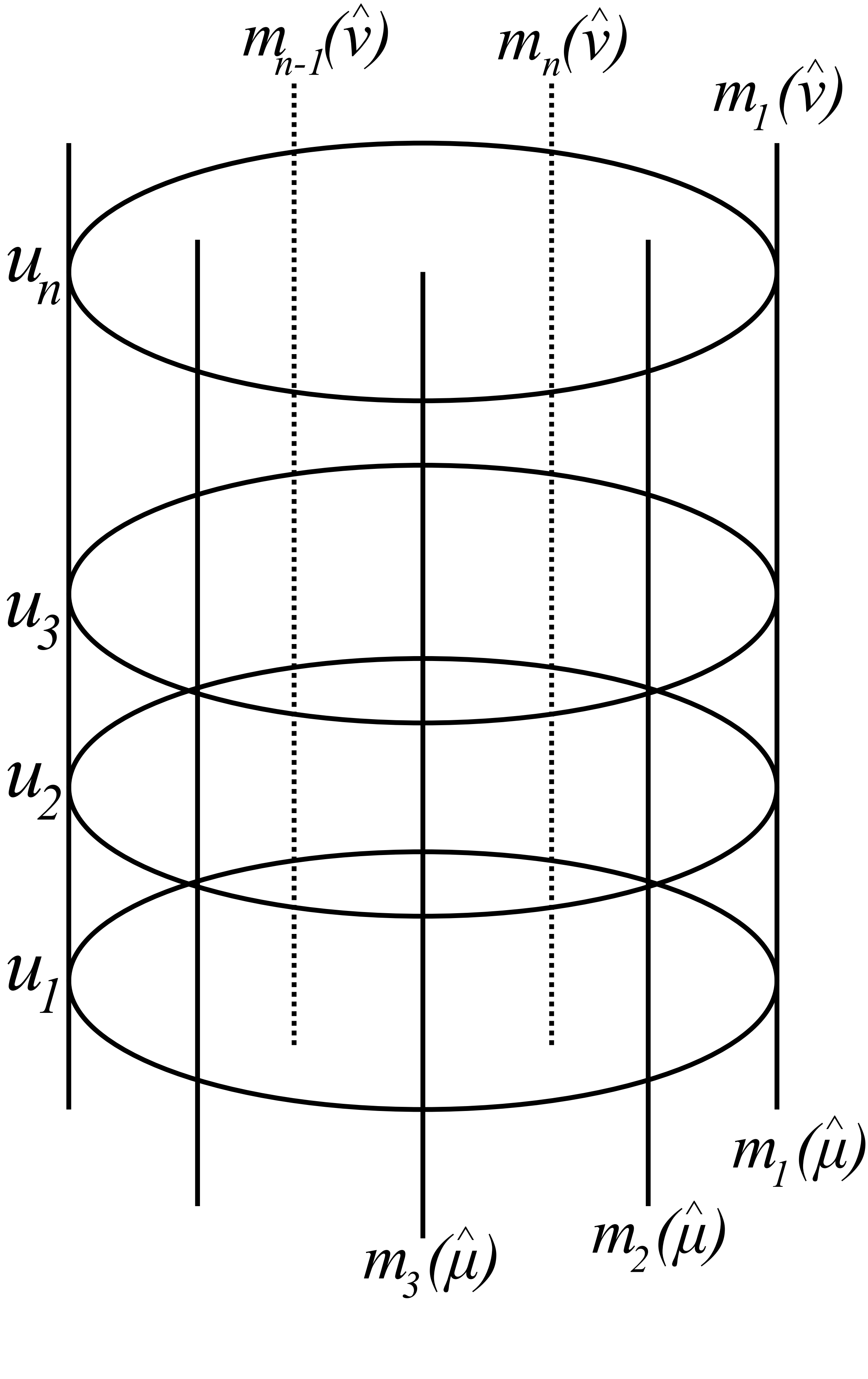

$m_{n-1}(\hat{v})$
$m_n(\hat{v})$
$m_1(\hat{v})$
u_n
u_3
u_2
u_1
$m_1(\hat{\mu})$
$m_3(\hat{\mu})$
$m_2(\hat{\mu})$